EXERCISE – IV HINTS & SOLUTIONS

Sol.1 I.f. = $e^{\int \ell n\, 2 dx} = (1/2)^x$

so $y(1/2)^x = \int 2^{(\sin x - x)}(\cos x - 1)\, \ell n\, 2\, dx$

$\sin x - x = t$ (let)

$\Rightarrow y.\left(\frac{1}{2}\right)^x = 2^{\sin x - x} + c$

$c = 0$ as $x \to \infty$ so $y = 2^{\sin x}$

Sol.2 $\frac{dy}{dx} = y + \int_0^1 y\, dx$

Let $A = \int_0^1 y\, dx$

$\frac{dy}{dx} = y + A \quad \Rightarrow \quad \frac{dy}{dx} - y = A$

If = $e^{\int -1.dx} = e^{-x}$

$y(e^{-x}) = A e^{-x} + c \quad \Rightarrow \quad y = A + c\, e^x$

$y = 1 \Rightarrow x = 0 \quad \Rightarrow c = 1 + A$

$y = A + (1 + A)\, e^x$

$A = \int_0^1 y\, dx \Rightarrow \quad A = \int_0^1 A + (1 + A)e^x dx$

$\Rightarrow A = \frac{e - 1}{3 - e}$

$y = \frac{e - 1}{3 - e} + \left(1 + \frac{e - 1}{3 - e}\right) e^x$

$\Rightarrow y = \frac{1}{3 - e}\,(2e^x - e + 1)$

Sol.3 Tangents at $y = f(x)$ & $y = \int_{-\infty}^{x} f(t)\, dt$

$T_1 : y - f(x) = f'(x)\,(X - x)$ .....(i)

$T_2 : Y - y_1 = f(x)\,(X - x)$ .....(ii)

$\{\because \frac{dy}{dx} = f(x)\}$

from (i) & (ii) x cordinates are equal & y = 0

$\Rightarrow x - \frac{f(x)}{f(x)} = x - \frac{y_1}{f(x)} \Rightarrow \frac{f(x)}{y_1} = \frac{f'(x)}{f(x)}$

Integrating : $\ell n\,(y_1) = \ell n\,(f(x).c) \Rightarrow y_1 = f(x).c$ ...(iii)

at x = 0 ; f(0) = 1

& $y_1 = \int_{-\infty}^{x} f(x)\, dx = \frac{1}{2} f(x)$

differentiating : $f(x) = 1/2\, f'(x) \Rightarrow \frac{f'(x)}{f(x)} = 2$

on integrating : $f(x) = c_1\, e^{2x}$: x = 0, f(0) = 1

so $f(x) = e^{2x}$ $\quad c_1 = 1$

Sol.4 $\frac{dy}{dx} + P(x)y = Q(x)$

(i) $u' + Pu = Q$

$v' + Pv = Q$

subtract $Q = \frac{uv' - vu'}{u - v}$

$\int \frac{u' - v'}{u - v} = \int -P$

$\ell n\,(u - v) = -\int P\, dx$

$u - v = e^{-\int P\, dx}$

IF = $e^{\int P\, dx} = \frac{1}{u - v}$

$y\left(\frac{1}{u - v}\right) = \int \frac{uv' - vu'}{(u - v)^2} dx$

$= \int \frac{d(v/u)}{\left(1 - \frac{v}{u}\right)^2}$

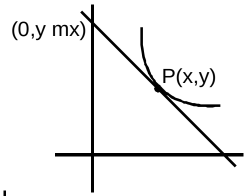
$y\left(\frac{1}{u - v}\right) = \frac{u}{u - v} + k$

$y = u + k\,(u - v)$

(ii) $\alpha = 1 + k, \quad \beta = -k$

$\alpha + \beta = 1$

Sol.5 $x^2 + m^2x^2 = 4$



$$\frac{dy}{dx} = \pm\sqrt{\frac{4-x^2}{x^2}}$$

solve by taking

+ve & –ve sign seprately

Sol.6 $xdy + ydx = \dfrac{ydx - xdy}{x^2+y^2}$

$\Rightarrow$ $d(xy) = -\,d(\tan^{-1}(y/x))$

$\Rightarrow$ $xy + \tan^{-1}(y/x) = c$

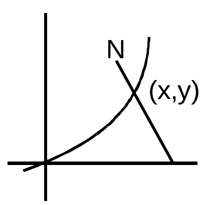
Sol.7 $-\displaystyle\int d\left(\frac{y}{x-y}\right) = \int \frac{dx}{2\sqrt{1-x^2}}$

$$\Rightarrow -\left(\frac{y}{x-y}\right) = \frac{1}{2}\sin^{-1}x + c$$

$c = 2 - \pi/4$ as $y = 2$ when $c = 1$

so $\dfrac{\sin^{-1}x}{2} + \dfrac{y}{x-y} = \dfrac{\pi}{4} - 2$

Sol.8 Equation of normal



$$Y - y = \frac{-1}{dy/dx}(X - x)$$

Mid point (h, k)

$$= \left(\frac{y\,dy/dx + 2x}{2}, \frac{y}{2}\right)$$

put in curve : $y.\dfrac{dy}{dx} - y^2 = 2x$

Let $y^2 = t$ & solve the L.D.E.

Sol.9 $\displaystyle\int_0^x t(f(x-t)dt = \int_0^x f(t)dt + \sin x + \cos x - x - 1$

Let $x - t = z$ $\Rightarrow$ $dt = -dz$

$$\int_0^x (x-z)f(z)\,dz = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$$

$$x\int_0^x f(z)\,dz - \int_0^x zf(z)\,dz = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$$

Use liebnitz

$$\int_0^x f(z)\,dz + x\,f(x) - x\,f(x) = f(x) + \cos x - \sin x - 1$$

again leibnitz

$f(x) = f'(x) - \sin x - \cos x$

$$\frac{dy}{dx} - y = \sin x + \cos x$$

$IF = e^{-x}$

$$y(e^{-x}) = \int e^{-x}(\sin x + \cos x)\,dx$$

$y(e^{-x}) = -e^{-x}\cos x + c$

$y = c\,e^x - \cos x$

$x = 0 \Rightarrow y = 0$

$c = 1$

$y = e^x - \cos x$

Sol.10 $\dfrac{dy}{dx} + y\dfrac{1}{(1-x^2)^{3/2}} = \dfrac{x+\sqrt{1-x^2}}{(1-x^2)^{3/2}}$

$$\text{I.F.} = e^{\int \frac{1}{(1-x^2)^{3/2}}dx}$$

Sol.11 $3x^2y^2\,dx + \cos(xy)\,dx - xy\sin(xy)\,dx + 2x^3y\,dy - x^2\sin(xy)\,dy = 0$

$d(x^3y^2) + \cos xy\,dx - x\sin(xy)\,d(xy) = 0$

$d(x^3y^2) + d(x.\cos(xy)) = 0$

integrate

$x^3y^2 + x\cos(xy) = c$

$x(x^2y^2 + \cos(xy)) = c$

Sol.12 $x(1 - x\,\ell n\,y).\dfrac{dy}{dx} + y = 0$

$$\Rightarrow y.\frac{dy}{dx} + x = x^2\log y$$

divding by $x^2 y$

$$\frac{1}{x^2}\frac{dx}{dy} + \frac{1}{xy} = \frac{1}{y}\log y$$

Let $\dfrac{1}{x} = t \Rightarrow -\dfrac{1}{x^2}\dfrac{dx}{dy} = \dfrac{dt}{dy}$

$$\Rightarrow -\frac{dt}{dy} + \frac{t}{y} = \frac{1}{y}\log y \Rightarrow \frac{dt}{dy} - \frac{t}{y} = -\frac{1}{y}\log y$$

If $\Rightarrow$ $e^{-\int \frac{1}{y}dy} = e^{-\log y} = 1/y$

sot .1/y = $\int -\frac{1}{y^2}\log y\,dy = \frac{1}{y}\log y - \int \frac{1}{y}.\frac{1}{y}\,dy$

$\Rightarrow$ t = log y + 1 + cy $\Rightarrow$ log ex + cy

$\Rightarrow \frac{1}{x}$ = log e y + cy $\Rightarrow$ x (log ey + cy) = 1

Sol.13 Equation of tangent at (x, y) for y = f(x)

Y – y = f ′(x) (X – x).

at x-axis point A = $\left(-\frac{y}{f'(x)} + x, 0\right)$

at y-axis point B = (0, –xf ′(x) + y)

given $\dfrac{-\frac{y}{f'(x)} + x + x}{2} = 0 \Rightarrow -y + 2xf'(x) = 0$ ...(i)

& $\frac{0+y}{2} = -x f'(x) + y \Rightarrow -2x f'(x) + y = 0$ ...(2)

on solving (1) & (2) curve is $y^2 = cx$

Sol.14 Let the curve is y = – f(x) & point P is (x, y)

so point A is (x, 0)

equation of tangent at P is

Y – y = f ′(x) (X – x) .....(1)

Length of $\perp^r$ from (x, 0) to tangent is ‘a’

so $\left|\frac{y}{(f'(x))^2 + 1}\right| = a$

on squaring : $y^2 = a^2 (f'(x))^2 + a^2$

$f'(x) = \pm\sqrt{\frac{y^2}{a^2} - 1}$ .....(2)

solve by taking +ve & –ve sign seprately also on y-axis, x = 0

& angle b/w curve & y-axis is $\frac{\pi}{2}$

so $\Rightarrow f'(x) = 0$ .....(3)

Sol.15 Let the curve is y = f(x) & point of tangent is (x, y)

Equation of tangent : Y – y = f ‘(x) (X – x) .....(1)

at y-axis, intercept = y-x f ′(x) .....(2)

subnormal = y . f ′(x) .....(3)

Slope of tangent = f ′(x) .....(4)

Subtangent = $\frac{y}{f'(x)}$ .....(5)

according to question :

$$\frac{(y - x f'(x))^2}{y.f'(x)} = \frac{x \cdot y}{(f'(x))^2\left(\frac{y}{f'(x)}\right)} \quad .....(6)$$

Solve equation (6) & for C, use point (1, 0)

Sol.16 Let at any instant t, x be the volume of water in reservoir A & y of that in B.

$\frac{dx}{dt} \times x \Rightarrow \frac{dx}{dt} = k, x \Rightarrow x = e^{K_t 1} e^{C_1}$ ....(1)

Similarly $\frac{dy}{dx} \times y = 1$ $y = e^{K_2 t} e^{C_2}$ ....(2)

Now at t = 0; x = 2y $\Rightarrow$ x/y = 2

from (1) & (2) $\frac{eC_1}{eC_2} = 2$ ....(3)

also at t = 1 ; $x = \frac{3}{2}y \Rightarrow \frac{x}{y} = \frac{3}{2}$

$\Rightarrow \frac{e^{K_1} e^{C_1}}{e^{K_2} e^{C_2}} = \frac{3}{2} \Rightarrow e^{K_1 - K_2} = 3/4$

Let at t = T : x = y $\Rightarrow \frac{x}{y} = 1$ then $\frac{e^{K_1 T} e^{C_1}}{e^{K_2 T} e^{C_2}} = 1$

$\Rightarrow e^{(K_1 - K_2)T} = 1/2 \Rightarrow (3/4)^T = \frac{1}{2} \Rightarrow T = \log 4/3^2$

Sol.17 $\frac{dm}{dt} = 10 - \left(\frac{m}{50 + t}\right)$

$\frac{dm}{dt} + \frac{m}{50 + t} = 10$

IF = $e^{\int \frac{dt}{50+t}} = 50 + t$

$m(50 + t) = \int 10(50+t)\,dt$

$m(50 + t) = 10\left(50t + \frac{t^2}{2}\right)$

$m = 5\left(\frac{100t + t^2}{50+t}\right)$

$= 5t\left(\frac{100+t}{50+t}\right)$

$= 5t\left(1 + \frac{50}{50+t}\right)$

at $t = 10$

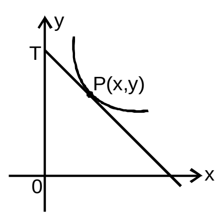
$m = 91\frac{2}{3}$

Sol.18 Assume two curves

$\frac{dy}{dx} = \frac{a_1x + a_2y}{a_3x + a_4y}$ & $\frac{dy}{dx} = \frac{b_1x + b_2y}{b_3x + b_3y}$

Solve this differential equation by putting $y = tx$
and then solve with $y = mx$ to get P_1 and P_2
& get the slope at P_1 & P_2

Sol.19 $Y - y = \frac{dy}{dx}(X - x)$



$OT = y - x\frac{dy}{dx}$

& $y - x\frac{dy}{dx} = Kx^3 \Rightarrow \frac{x\,dy - y\,dx}{x^2} = -\,kx\,dx$

on integrating

$\frac{y}{x} = -\frac{-kx^2}{2} + C$

Sol.20 **(i)** $y = ax^2 \Rightarrow \frac{dy}{dx} = 2ax$

$-\frac{dx}{dy} = 2ax$

$-\frac{dx}{dy} = \frac{2y}{x} \qquad \Rightarrow -x\,dx = 2y\,dy$

integrate $\qquad \Rightarrow x^2 + 2y^2 = k$

(ii) $\cos y = ae^{-x}$

$-\sin y\frac{dy}{dx} = -ae^{-x}$

$\sin y\frac{dy}{dx} = \cos y$

$\sin y\left(-\frac{dy}{dx}\right) = \cos y$

$-dx = \cot y\,dy$

$\ln(\sin y) = -x + c$

$\sin y = ke^{-x}$

(iii) $x^k + y^k = a^k$

$kx^{k-1} + ky^{k-1}.\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x^{k-1}}{y^{k-1}}$

$-\frac{dx}{dy} = -\frac{x^{k-1}}{y^{k-1}}$

$\frac{dx}{x^{k-1}} = \frac{dx}{y^{k-1}}$

After intergrating

$\frac{1}{x^{k-2}} - \frac{1}{y^{k-2}} = \frac{1}{c^{k-2}}$ it k = 2

(iv) $x^2 - y^2 = a^2$

$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\tan 45° = \left|\frac{\frac{dy}{dx} - \frac{x}{y}}{1 + \frac{x}{y}\frac{dy}{dx}}\right|$

+ve : $1 + \frac{x}{y}\frac{dy}{dx} = \frac{dy}{dx} - \frac{x}{y}$

$\frac{dy}{dx} = \frac{y+x}{y-x}$

homogenus Eq^n put $y = tx$

after integration $x^2 - y^2 + 2xy = c$

–ve : $x^2 \ \ y^2 - 2xy = c$